Volatility and Dividends II: Consistent Cash Dividends

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Abstract

We discuss a framework for time-homogeneous equity stock price models with a consistent dividend process such that at any point, conditional on the state variables of the model, short-term implied dividends are "cash-like" (constant) and long-term dividends are "proportional".

First, we show that in general any stock price process S with drift μ which pays dividends $(\Delta_j)_j$ on $t_1 < \cdots$ can be written as follows: let $R_t = \exp(\int_0^t \mu_s\, ds)$ and $S_0^* = S_0 - \sum_j R_{t_j}^{-1}\, \mathbb{E}[\Delta_j]$, then S can be written as

$$\frac{S_t}{R_t} = S_0^* X_t + \sum_{j: t_j > t} \frac{1}{R_{t_j}} \mathbb{E}_t[\Delta_j]$$

where X is a non-negative (local) martingale.

With this in mind we propose a model where X is a local-volatility process $dX_t = X_t \sigma_t(X_t)\, dW_t$, and where each dividend is modelled as $\Delta_t := \mathbb{E}[\Delta_j]\, Y_{t_j}$ for a mean-reverting process Y of the type

$$dY_t = \kappa(\alpha X_t + \hat{\alpha} - Y_t)\, dt + \Sigma_t(Y_t) dB_t \ ,$$

with W correlated to B.

This model has the desired "consistent" dynamics of short-term cash and long-term proportional dividend behaviour, while allowing efficient calculation of spot, forward and dividend swaps as a function of (X_t, Y_t) inside a Monte Carlo simulation or Finite Difference scheme.

This note formalizes results presented in 2012 at Global Derivatives [4]. The present note lacks some illustrations but is otherwise complete.

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1 Introduction

In this article, we present a modelling approach for stochastic dividends. The key characteristic we want to capture with our model is the simple economic fact that the level of implied short-term dividends as a function of company earnings are much more certain than implied long-term dividends: it is much easier to predict the dividend income from a typical equity index basket for next year then, say, for the period between 10Y and 11Y from now. (Note that the market's "expectation" is usually known for most indices since future dividends can be traded using futures or swaps.) Typically, implied long-term dividend payments are thought to behave proportionally to the overall level of the equity market, while short-term implied dividends are thought to be practically known and therefore nearly constant, or "cash-like". In other words, short-term dividends are much less volatile than long-term dividends.

The aim of this note is to discuss dividend analytics. We do not discuss calibration of our model in detail. We merely note that standard methods such as particle calibration or forward-PDE methods might be used for the proposed two-factor case.

Consistent Cash Dividends

The simplest approach to handle such a scenario is to take today's implied dividend curve and to blend it from "cash" to "proportional" dividends over time with a time-dependent blending function. For example, assume we are in January 2014. We may choose to model dividends up to 2017 as cash, and then to blend them into proportional dividends, with full proportional dividends starting after 2019. In such a setup, an option on the dividends paid between 2Y (2016) and 3Y (2017) would have no time value. The mathematical framework for this approach is described in detail in Buehler [3].

While this method yields practical results and is easy to implement across a wide range of underlying dynamics, it suffers on a methodological level from its time-dependent nature. Basically, what happens is that the dividends are always proportional after 2017. That means that if we priced a forward-starting option on realized dividends, struck in 2017 for the dividends paid between then-2Y (2019) and then-3Y (2020), then this option would have quite a bit of time-value since the dividends "in the model" are assumed to be stochastic and therefore move as an average over the stock price over the respective period. We call this "inconsistent" behaviour: in the absence of additional information, why would a forward started option for the dividends between 2Y and 3Y priced tomorrow be much different than the same option priced in a few year's time.

"Consistent" models, on the other hand, are models where the principal dynamic properties of the underlying market diffusion are time-independent or, practically speaking, where the dynamics of the model conditional on a future state of the model are similar to the model today: in our case, short-term dividends are cash-like and have very low implied volatility, while long-term dividends remain volatile.[1] The difference in the prices of forward started options are about levels of implied volatility, not about no volatility vs. a lot of volatility.

General Representation of Dividend Paying Stocks

Our modelling approach is based on the observation that every dividend paying stock can be written under the risk-neutral measure as a sum of a non-negative (local) martingale X, representing the "inner value" of the stock, plus the discounted expected value of all future dividends: assuming random dividends $(\Delta_j)_j$ are paid at $t_1 < \cdots$, we will show that we can write the appropriately discounted stock price $\tilde{S}$ in terms of a local martingale as X as

$$\tilde{S}_t = S_0^* X_t + \sum_{j:t_j>t} \mathbb{E}_t[\tilde{\Delta}_j] \quad \text{with} \quad S_0^* = S_0 + \sum_j \mathbb{E}_0[\tilde{\Delta}_j] \ ,$$

where $\tilde{\Delta}_j$ denotes the value of the jth dividend "discounted" with the funding rate of the equity. Note that for such a model, in order to compute spot, we need to have an efficient way of computing $\mathbb{E}_t[\tilde{\Delta}_j]$.

We therefore propose to model directly X (as a local volatility process) and the random dividends (as functions of a process Y which mean-reverts around X). The specification is such that spot, forwards and the value of dividend swaps, conditional on a future point (X_t, Y_t) can be computed efficiently.

Long Story Short

To cut things short, we are looking for a model where, conditional on a future state of the world, short-term dividends are cash-like (ie, have little volatility)

[1] We use the notion of "consistency" here in the same sense as used by Filipovic in for rates term structure models [6] and by Buehler in the context of variance swap term structure models [2].

while long-term dividends are proportional (ie, highly correlated to the stock price).

To our knowledge, no such model has been presented to the public so far. The results of this paper have been presented on Global Derivatives 2012 [4].

2 Model Setup

Let S be a stock which pays random non-negative dividends $\Delta = (\Delta_j)_j$ with $j = 1, \ldots$ at dividend dates $0 < t_1 < t_2 < \cdots$. We denote by $\delta_j := \mathbb{E}[\Delta_j]$ the initial expected values under the risk-neutral measure $\mathbb{P}$. We further assume for simplicity that rates and the equity funding spread are deterministic. We denote by μ the overall funding rate of the equity and set $R_t := \exp(\int_0^t \mu_s \, ds)$ to define the "discounted" quantities $\tilde{S}_t := S_t / R_t$, $\tilde{\Delta}_j := \Delta_j / R_{t_j}$ and $\tilde{\delta}_j := \mathbb{E}[\tilde{\Delta}_j]$.

Finally, we also denote by $S_0^* := S_0 - \sum_j \tilde{\Delta}_j$ the "inner spot" of S, which represents the part of the equity value S_0 which is not due to the present value of future expected dividends. In order to avoid negative forwards, S_0^* must not be negative.

Theorem 2.1 (Generalized Representation of Dividend Paying Stocks) *For any non-negative stock price process S which pays Δ, there exists a non-negative (local)[2] martingale X such that*

$$\tilde{S}_t = S_0^* X_t + \sum_{j: t_j > t} \mathbb{E}_t[\tilde{\Delta}_j] \ . \tag{Eq:X}$$

Reversely, for any non-negative (local) martingale X, the stock price S defined by (Eq:X) is non-negative and pays dividends Δ.

The discounted forward of S is given by taking expectations[3] as

$$\tilde{F}_t = S_0^* + \sum_{j: t_j > t} \tilde{\Delta}_j \ . \tag{1}$$

As in [3] for the affine dividend case, we want to stress that the form (Eq:X) is <u>not</u> a modelling assumption, but a consequence of S paying dividends Δ and staying positive; any other non-equivalent representation will require imposing additional constraints on the local martingale part to ensure non-negativity of the stock price process. As an example, consider the following:

Corollary 2.1 *Let*

$$Z_t := X_t + \frac{1}{S_0^*} \sum_j \mathbb{E}_t[\tilde{\Delta}_j] \ . \tag{Def:Z}$$

[2] The restriction to a "local" martingale is relevant for the existence of "no free lunch with vanishing risk", a form of no-arbitrage which guarantees the existence of, indeed, just a local martingale for any tradable asset; see [5] and the discussion in section 5.

[3] Strictly speaking, this is only true up to any horizon for which X has true martingale property.

Then, Z is a non-negative (local) martingale and we have

$$\tilde{S}_t = S_0^* Z_t - \sum_{j:t_j \le t} \tilde{\Delta}_j \ . \tag{Eq:Z}$$

The SDE for S can then be written as

$$dS_t = \mu S_t \, dt + S_0^* R_t \, dZ_t - \Delta_t \, dt \ .$$

While it is tempting from a modelling perspective to specify Z rather than X in (Eq:Z), this is more complicated as non-negativity of Z does not simply imply non-negativity of S; indeed, for any Z we specify, the associated process X via (Def:Z) must remain non-negative. We therefore propose to model a dividend paying stock using (Eq:X), rather than (Eq:Z).

As noted in [3], the main drawback of this (correct) approach is that the dynamics of S_T depend on the dynamics of dividends past T. Practically, that means that an option with maturity T has sensitivity to implied dividends past its maturity.

Both the theorem and its corollary are proved in a general setting in section 5.

Standard assumptions on the dividends Δ^j are:

- **Cash Dividends**: simply $\Delta_j = \delta_j$.
- **Proportional Dividends**, i.e. $\Delta_j \sim X_{t_j}$.

  This is the classic setting in the (extended) Black&Scholes model [1]: assume one is given a forward curve $(F_t)_t$ and define $S_t := F_t X_t$ for a martingale X. Then, $\Delta_j = S_{t_j -} \delta_j$. Typically, this is written in terms of proportional discrete dividend rates η_j as $\delta_j = 1 - e^{-\eta_j}$ such that $S_{t_j} = S_{t_j -} e^{-\eta_j}$.

Both cases are covered in depth in [3], including the case with credit risk.

2.1 Cash Yield Framework

As a first step in this article, we reformulate for convenience of notation the above discrete dividend setting to the more general case of a cash yield framework, where $\Delta = (\Delta_t)_{t \ge 0}$ is a non-negative process and the stock price is given consistently with (Eq:X) as

$$\tilde{S}_t = S_0^* X_t + \int_t^\infty \mathbb{E}_t[\tilde{\Delta}_s] \, ds \quad \text{with} \quad S_0^* := S_0 - \int_0^\infty \tilde{\delta}_s \, ds \ , \tag{2}$$

where $\delta_t := \mathbb{E}[\Delta_t]$ as before. We assume that S_0^* is non-negative, which implies that the expectation for all dividends is bounded. The forward in this formulation is given as

$$\tilde{F}_t = S_0^* + \int_t^\infty \tilde{\delta}_s \, ds \ . \tag{3}$$

We recover (Eq:X) by setting $\Delta_{t_j} := \Delta_j \boldsymbol{\delta}_{t_j}(dt)$ and $\Delta_t := 0$ for all $t \notin \{t_j\}_j$. (Here, $\boldsymbol{\delta}_{t_j}(\cdot)$ denotes the Dirac measure in t_j, i.e. $\int \boldsymbol{\delta}_t(s) \, ds = t$.)

3 Consistent Cash Dividends

The main idea we want to present in this paper is an improvement over the affine dividend approach to modeling dividends discussed in [3]. We present a model where at any point in time short term implied dividends have low volatility (are "cash"-like), while long-term dividends have higher volatility and are strongly correlated to the equity.

Our approach is to define the random dividends Δ_t as a function of a secondary driving process $(Y_t)_t$ which implements the desired behaviour and which, conditional on (X_t, Y_t), still lets us efficiently compute beyond spot also implied dividends and forward forwards,

$$\mathbb{E}_t\left[\,\Delta_s\,\right] \quad \text{and} \quad \mathbb{E}_t\left[\,S_s\,\right] \qquad (t \le s)\ ,$$

respectively.

Consistency of the dividend curve translates mathematically in the requirement of similarity in distribution, i.e.

$$\mathbb{E}\left[\,f(\Delta_{t+\tau})\,\middle|\,(X_t, Y_t) = (x, y)\,\right] \cong \mathbb{E}\left[\,f(\Delta_{u+\tau})\,\middle|\,(X_u, Y_u) = (x, y)\,\right]$$

for any measurable function f with compact support. The "$\cong$" becomes an equality if the model itself has homogeneous (constant, time-independent) parameters.

3.1 Model Specification

In the light of (2), let X be the solution to

$$\frac{dX_t}{X_t} = \sigma_t(X_t)\,dW_t \qquad (X_0 = 1) \tag{4}$$

with a suitably regular local volatility σ. The latter will be calibrated to the market prices of equity options. We define the dividend-driving process Y as a mean-reverting process of the form

$$dY_t = \kappa(\alpha X_t + \hat{\alpha} - Y_t)\,dt + \Sigma_t(Y_t) dB_t \qquad (Y_0 = 1) \tag{5}$$

with $\alpha \in [0, 1]$, $\hat{\alpha} = 1 - \alpha$ and where Σ guarantees that $Y > 0$, for example $\Sigma(y) = \Sigma y$. We specifically allow the "martingale case" $\kappa = 0$. Adding jumps is discussed in section 3.3.

The main characteristic of Y is that it mean-reverts around the martingale

$$X_t^a := \alpha X_t + \hat{\alpha}\ .$$

For $\alpha = 1$ it mean-reverts around the pure equity driver X. For a small Σ that means that Y becomes a exponential running average of the spot price. For $\alpha = 0$, on the other hand, the two processes X and Y are only related through their correlation. We comment on several specific configurations below.

For $u > t$ the process Y satisfies

$$Y_u = e^{-\kappa(u-t)}Y_t + \kappa \int_t^u e^{-\kappa(u-s)} X_s^a \, ds + \int_t^u e^{-\kappa(u-s)} \Sigma_s(Y_s) \, dB_s \qquad (6)$$

and therefore

$$\mathbb{E}_t \left[Y_u \right] = e^{-\kappa(u-t)}Y_t + \left(1 - e^{-\kappa(u-t)}\right) X_t^\alpha \ . \qquad (7)$$

In particular, we have $\mathbb{E}[Y_t] \equiv 1$, which allows us to define the actual dividend payment Δ_j simply as the product of Y_t and today's expected dividend δ_t as

$$\tilde{\Delta}_t := \tilde{\delta}_t Y_t \ . \qquad (8)$$

For $0 \leq t \leq s$ this gives us

$$\mathbb{E}_t[\tilde{\Delta}_s] = \tilde{\delta}_s \left(X_t^\alpha + (Y_t - X_t^\alpha) e^{-\kappa(s-t)} \right) \ , \qquad (9)$$

and therefore

$$\tilde{S}_t = S_0^* X_t + \int_t^\infty \tilde{\Delta}_s \left(X_t^\alpha + (Y_t - X_t^\alpha) e^{-\kappa(s-t)} \right) \, ds \ , \qquad (10)$$

or, in discrete form,

$$\tilde{S}_t \equiv S_0^* X_t + \sum_{j:t_j > t} \tilde{\Delta}_j \left(X_t^\alpha + (Y_t - X_t^\alpha) e^{-\kappa(t_j - t)} \right)$$

(we will continue using the "$\equiv$" operator to translate the integral into the discrete from). All quantities are straight forward to compute given (X_t, Y_t), as we will show below.

Model Choices

The model covers a range of cases:

- **Yield Dividends**: with very strong mean reversion $\kappa \uparrow \infty$, $\Sigma \equiv 0$ we get $Y \to X$.
- **Cash Dividends**: are the case were $\Sigma \equiv 0$ and $\alpha = 0$, in which case we simply have $Y \equiv 1$, and the model becomes a fixed cash dividend model.
- **Running Average**: in case $\Sigma \equiv 0$, $\kappa < 1$ (say) and $\alpha = 1$, the process Y is basically the rolling exponential average of X,

  $$Y_t = Y_0 + \int_0^t e^{-\kappa t} X_s \, ds$$

  This in itself is an attractive model since it allows for a good interpretation of the dividend policy as a function of average past stock performance.

- **Running Average with Noise**: adding an additional uncorrelated noise $\Sigma > 0$, $\rho = 0$ introduces additional uncertainty around the actual dividends. Schematically, it gives us

$$Y_t = Y_0 + \int_0^t e^{-\kappa t} X_s \, ds + \varepsilon_t$$

  This is the author's favorite use of the model.

- **Martingale Case**: by setting $\kappa = 0$ and $\Sigma = \sigma$, the dividend-driving process Y becomes itself a martingale. If correlated highly with X, this can be interpreted as a reasonable model for an equity. In this case, there are fast approximations for the price of options on dividends (option pricing on averages).

- **Affine Dividends**: setting $\alpha = 0$ turns Y into a standard mean-reverting process which drives the dividends. For the specific case $\Sigma(y)y \equiv \Sigma\sqrt{y}$ we obtain the CIR square-root process. This means in particular that we have a closed form Fourier-solution for pricing options on dividends.

3.2 Efficient Representation

Let us now transform the basic equation (10) for S_t into a concise representation in terms of numerically readily available quantities. Let

$$\boxed{\tilde{D}_t^\kappa := \int_0^t e^{-\kappa s} \tilde{\delta}_s \, ds \equiv \sum_{j: t_j \le t} e^{-\kappa t_j} \tilde{\delta}_s} \qquad \text{(Def:}\tilde{D}\text{)}$$

and define $\tilde{C}_t^\kappa := e^{\kappa t} \int_t^\infty e^{-\kappa s} \tilde{\delta}_s \, ds$, which we can write in terms of $\tilde{D}$ as

$$\boxed{\tilde{C}_t^\kappa = e^{\kappa t} \left(\tilde{D}_\infty^\kappa - \tilde{D}_t^\kappa \right) \ .} \qquad \text{(Def:C)}$$

A term structure of $\tilde{C}$ can be computed easily up front before any numerical simulation takes place.

Using $\tilde{C}$ allows us writing (10) as the very concise

$$\boxed{\tilde{S}_t = S_0^* X_t + (\tilde{C}_t^0 - \tilde{C}_t^\kappa) X_t^\alpha + \tilde{C}_t^\kappa Y_t \ .} \qquad \text{(Eq:S)}$$

Note that taking expectations implies that the (discounted) forward has for all t the form

$$\boxed{\tilde{F}_t = S_0^* + \tilde{C}_t^0 \ .} \qquad \text{(Eq:F)}$$

Specifically, $S_0 = S_0^* + \tilde{C}_0^0$. In the specific case $\alpha = 1$ equation (Eq:S) simplifies to

$$\tilde{S}_t \stackrel{\alpha \equiv 1}{=} (S_0^* - \tilde{C}_t^\kappa) X_t + \tilde{C}_t^\kappa Y_t \ .$$

It follows from (3) that

$$d\tilde{F}_t = -\tilde{\delta}_t dt \ , \tag{11}$$

which with (Def:C) gives

$$\tilde{C}_t^\kappa = -e^{\kappa t}\int_t^\infty e^{-\kappa s} d\tilde{F}_s \ .$$

This representation is useful if our pricing environment provides directly a forward curve, rather than explicitly a list of dividends.

Dividend Swaps

The time-t expectation of future dividends over $[T, T+x]$ with $T \geq t$ is formally given as

$$D_t(T,x) := \int_T^{T+x} \mathbb{E}_t[Y_s]\delta_s ds \equiv \sum_{j:T<t_j\leq T+x} \mathbb{E}_t[Y_{t_j}]\,\delta_{t_j} \ . \tag{12}$$

This is essentially the time-t value of a divdend swap paying dividends between T and $T + x$. Specifically, dividends here are not discounted.

Using (9) and (11) we get

$$\begin{aligned} D_t(T,x) &= \int_T^{T+x} \Big(e^{-\kappa(s-t)}(Y_t - X_t^\alpha) + X_t^\alpha \Big)\,\delta_s\,ds \\ &= (Y_t - X_t^\alpha)e^{\kappa t}\int_T^{T+x} e^{-\kappa s}\delta_s\,ds + X_t^\alpha \int_T^{T+x}\delta_s\,ds \ . \end{aligned}$$

Define therefore

$$\boxed{D_t^\kappa := \int_0^t e^{-\kappa s}\delta_s\,ds \equiv \sum_{j:t_j\leq t} e^{-\kappa t_j}\delta_{t_j} \ .} \tag{Def:D}$$

This means the value of the dividend swap is given as

$$\boxed{D_t(T,x) = (Y_t - X_t^\alpha)e^{\kappa t}\left(D_{T+x}^\kappa - D_T^\kappa\right) + X_t^\alpha\left(D_{T+x}^0 - D_T^0\right)} \tag{Eq:DS}$$

Again, D can be computed easily up front before any numerical simulation takes place.

3.3 Jumps

A rather natural extension is to add jumps to the equity process. We have two ways to introduce jumps: first, as part of X. A jump will then indirectly affect Y via the latter's mean-reversion to the former. Secondly, we can also add jumps to Y itself.

As an example which covers both cases, assume that N and M are Poisson-processes with intensities λ and ω, respectively. Let $(\xi_i)_i$, $(\psi_i)_i$ and $(\phi_i)_i$ be iid sequences of normal variables. Define first the stock price driver as

$$\log X_t := \int_0^t \sigma_t(X_{t-})\, dW_t - \frac{1}{2}\sigma_t^2(X_{t-})\, dt + \sum_{i=1}^{N_t} \xi_i - \mathbb{E}[e^{\xi_1} - 1]\lambda\, dt\ .$$

Assuming a reasonably behaved local volatility σ, the process X is again a martingale.

Now define the martingales $\mathcal{M}$ and $\mathcal{N}$ as

$$\log \mathcal{N}_t := \sum_{i=1}^{N_t} \psi_i - \mathbb{E}[e^{\psi_1} - 1]\lambda\, t$$

and

$$\log \mathcal{M}_t := \sum_{i=1}^{M_t} \phi_i - \mathbb{E}[e^{\phi_1} - 1]\omega\, t\ .$$

The dividend driver is then defined as

$$Y_t = Y_0 + \kappa \int_0^t (\alpha X_{s-} + \hat{\alpha} - Y_{s-})\, ds + \int_0^t \Sigma_s(Y_{s-}) dB_s + \int_0^t Y_{s-}\, d(\mathcal{N}_s + \mathcal{M}_s)$$

with again

$$\mathbb{E}_t[Y_u] = e^{-\kappa(u-t)} Y_t + \left(1 - e^{-\kappa(u-t)}\right) X_t^\alpha$$

as in (7).

In other words, with this model specification all the previous analytics concerning expected dividends still apply.

4 Handling the Forward Curve

The key point about representation (Eq:S) above is that once C is known, we can compute S_t and, as we will see, associated quantities efficiently. Of course, if we truly have a projected dividend stream $(\delta_t)_t$ out to infinity, then C is given simply by (Def:C). However, in practise, we will usually observe dividends only up to some finite future point in time and will have to rely on projection (i.e. extrapolation) methods beyond that point. A basic requirement for such any projection is of course that the forward remains positive for all finite horizons.

The next sections are therefore dedicated to the calculation of C under various assumptions on the forward curve, projection and interpolation. We start with two simple but illustrative examples with parametric forward curves which show how the analytics play out in a simple set up. We then use these ideas to address the question of how to construct C efficiently in practical applications.

Logistically, we will in each case comment on the base assumptions, then construct C, and then show how dividend swaps are priced.

4.1 Toy Model I: Constant Proportional Dividend Yield

As a first example, we consider the case where the initial forward curve at time 0 is given using a constant proportional dividend yield η and a constant funding rate μ, i.e. $F_t = S_0 e^{(\mu-\eta)t}$. In this case, the expected cash dividends are via (11) given as $\tilde{\delta}_t = -\frac{d}{dt}\tilde{F}_t = \eta\tilde{F}_t$, and therefore $\delta_t = \eta F_t$. Using (Def:C) yields

$$\tilde{C}_t^\kappa = S_0 \eta e^{\kappa t} \int_t^\infty e^{-(\eta+\kappa)s}\, ds = S_0 e^{-\eta t} \frac{\eta}{\eta+\kappa} \ . \tag{13}$$

For every positive yield η we have $\tilde{C}_0^0 = S_0$. By (Eq:F) we have then $S_0^* = S_0 - \tilde{C}_0^0 = 0$, i.e. that the entire spot value of the equity is given by the discounted value of future dividends. This is too extreme in our view.

To alleviate this, we introduce an additional degree of freedom by applying the dividend yield only to a fraction $q \in [0,1]$ of the forward, i.e.

$$\tilde{F}_t = S_0 \left(q + \hat{q} e^{-\eta t} \right) \quad (\hat{q} := 1-q).$$

This way we retain with S_0^* as some "inner value" of the equity which is not explained by the present value of future dividend expectations; for most technology stocks, that inner value seems to be close to 1. The expected dividends are then $\delta_t = \hat{q}\eta F_t$.

With these assumptions we obtain

$$\tilde{C}_t^\kappa = \hat{q} S_0 e^{-\eta t} \frac{\eta}{\eta+\kappa} \ . \tag{14}$$

Dividend Swaps

The prices of dividend swaps under a constant dividend yield are given as follows:

$$\begin{aligned} D_t(T,x) &:= \int_T^{T+x} \mathbb{E}_t\left[\, Y_s \,\right] \hat{q}\eta F_s \, ds \\ &= \hat{q}\eta F_t \int_T^{T+x} e^{(\mu-\eta)(s-t)} \left(e^{-\kappa(s-t)} (Y_t - X_t^\alpha) + X_t^\alpha \right) ds \\ &= \hat{q}\eta F_t \left\{ (Y_t - X_t^\alpha) e^{-a_1(T-t)} \frac{1-e^{-a_1 x}}{a_1} + X_t^\alpha e^{-a_2(T-t)} \frac{1-e^{-a_2 x}}{a_2} \right\} \end{aligned}$$

with $a_2 = -\mu + \eta$ and $a_1 = a_2 + \kappa$.

We also have

$$D_t^\kappa \overset{\text{(Def:D)}}{=} \hat{q} S_0 \eta \int_0^t e^{(\mu-\eta-\kappa)s}\, ds = \frac{1-e^{at}}{a} \ .$$

with $a := -\mu + \eta + \kappa$, which confirms the previous calculation via (Eq:DS).

4.2 Toy Model II: Constant Cash Yield

Our second example is given by assuming that the implied dividends are given as a constant cash yield $\delta_t \equiv \delta$ with again a constant funding rate. The discounted value of future expected dividends is therefore

$$\tilde{C}_t^k = \delta e^{\kappa t} \int_t^\infty e^{-(\mu+\kappa)s}\, ds = e^{-\mu t} \frac{\delta}{\mu+\kappa} \ .$$

Using again the relation $S_0 = S_0^* + C_0^0$ we see that this is well defined as long as $0 \le \delta/\mu \le S_0$. This simply means that the stock cannot pay in expectation a higher cash dividend rate than $S_0\mu$ for eternity. Similar in spirit to the above discussion, we may therefore not want to specify δ directly but instead chose to define S_0^* via a fraction $q \in [0,1]$ of the allowed maximum S_0. With $\hat{q} = 1-q$ the equivalent dividend rate is then given as $\delta = S_0\mu\hat{q}$.

The forward in this approach is given as

$$\tilde{F}_t = S_0 - \int_0^t e^{-\mu s}\delta\, dt = S_0 - S_0\mu\hat{q}\, \frac{1-e^{-\mu t}}{\mu} = S_0\left(q + \hat{q}e^{-\mu t}\right) \ , \qquad (15)$$

and $\tilde{C}^\kappa$ is given as

$$\tilde{C}_t^k = S_0\hat{q}e^{-\mu t}\frac{\mu}{\mu+\kappa} \ . \qquad (16)$$

Dividend Swaps

The prices of dividend swaps under a constant dividend rate δ are given as

$$\begin{aligned} D_t(T,x) &= \delta \int_T^{T+x} \mathbb{E}_t\left[\, Y_s \,\right] ds \\ &= \delta \int_T^{T+x} \left(e^{-\kappa(s-t)}(Y_t - X_t^\alpha) + X_t^\alpha \right) ds \\ &= \delta \left\{ (Y_t - X_t^\alpha) e^{-\kappa(T-t)} \frac{1-e^{-\kappa x}}{\kappa} + x X_t^\alpha \right\} \ . \end{aligned}$$

We also have

$$D_t^\kappa \stackrel{\text{(Def:D)}}{=} \int_0^t e^{-\kappa s}\delta_s\, ds = \delta \begin{cases} \frac{1-e^{-\kappa t}}{\kappa} & (\kappa > 0) \\ t & (\kappa = 0) \end{cases} \ ,$$

which confirms the previous calculation via (Eq:DS).[4]

[4] Dividing by $\hat{q}$ and with obvious notation we have

$$\begin{aligned} (*) &= e^{\kappa t}\left(D_{T+x}^\kappa - D_T^\kappa\right)(Y_t - X_t^\alpha) + \left(D_{T+x}^0 - D_T^0\right) X_t^\alpha \\ &= e^{\kappa t}\left(\frac{1-e^{-\kappa(T+x)}}{\kappa} - \frac{1-e^{-\kappa T}}{\kappa} \right)(Y_t - X_t^\alpha) + (T+x-T)\, X_t^\alpha \\ &= e^{-\kappa(T-t)}\left(\frac{1-e^{-\kappa x}}{\kappa}\right)(Y_t - X_t^\alpha) + x X_t^\alpha \end{aligned}$$

Our presentation [4] contains several illustrations of this toy model.

4.3 Practical Forward Curve Analytics

The previous two examples were toy models which we introduced to make the reader familiar with the calculus around implied dividend curves in our model. We now want to focus on practical implementation.

The main tool is $\tilde{C}$ as defined by (Def:C). In practise, of course, we do not have dividends from the market predicted out to infinity. Let us assume we have tradable or otherwise reliable dividend expectations for ex-dividend dates $0 < t_1 < \cdots < t_N = T$. At T, we have $\tilde{F}_T = S_0 - \sum_{j=1}^{N} \tilde{\delta}_j$, and also

$$\tilde{F}_T \overset{\text{(Eq:F)}}{=} S_0^* + \tilde{C}_T^0 \ .$$

Basically, we have one degree of freedom: how much of the forward value at T can be attributed to the "inner value" of the equity vs. the value of the remaining dividends, $\tilde{C}_T^0 = \sum_{j:j>N} \tilde{\delta}_j$.

4.3.1 Extrapolation with Zero Dividends

The most trivial choice is obviously to assume there are no further dividends beyond T, and therefore to set $S_0^* = F_T$. All further analytics are straight forward.

4.3.2 Extrapolation with a Constant Cash Yield

We may alternatively assume that $S_0^* = q\tilde{F}_T$ for a $q \in [0,1]$, such that $\tilde{C}_T^0 = \hat{q}\tilde{F}_T$ with $\hat{q} = 1 - q$. In order to compute $\tilde{C}_T'^{\kappa}$, we may assume that the implied dividends beyond T are given by a constant cash yield δ_*, i.e.

$$\tilde{C}_T'^{\kappa} = e^{\kappa T} \int_T^{\infty} \delta_* e^{-(\kappa+\mu_*)s} \, ds = e^{-\mu_* T} \delta_* \ ,$$

where μ_∞ is a terminal drift for the asset, e.g. $\mu_* = \mu_T$. Hence,

$$\delta_* = e^{\mu_* T} \hat{q} \tilde{F}_T \ . \tag{17}$$

4.3.3 Extrapolation by Repetition

A practical method of extrapolating dividends is to assume that every year we do not have an implied market for might reasonably be just like the previous, known, one.

To that end let $K := \max\{k : t_N - t_k < 1\}$, i.e. $t_{K+1} < \cdots < t_N$ represent the dates for the last full year of available market expectations for dividends. Let $L := N - K$ and define the dividend dates beyond t_N as $t_{K+\ell+iL} := t_{K+\ell} + i$ for $\ell = 1, \ldots, L$ and $i = 1, \ldots$, with expected dividend $\delta_{K+\ell+iL} := \delta_{K+\ell}$.

We again fix a terminal equity forward rate μ_*.

Then,

$$\begin{aligned}\tilde{C}_T^\kappa &= e^{\kappa t}\sum_{i=0}^{\infty}\sum_{\ell=1}^{L} e^{-(\mu_*+\kappa)(t_{K+\ell}+iL)}\delta_{K+\ell} \\ &= e^{\kappa t}\big(\tilde{D}_{t_N}^\kappa - \tilde{D}_{t_K}^\kappa\big)\big(1-e^{(\mu_*+\kappa)L}\big)\end{aligned}$$

Here, $\tilde{C}_T^0$ is constrained by $\tilde{C}_T^0 \le S_0^*$.

5 General Representation of Dividend Paying Stocks

This section extends the structural results on the fundamental representation of dividend paying stocks of [3] for affine dividends to the stochastic dividend case. This section is of purely academic interest and self-contained.

Theorem 5.1 (Generalization of Theorem 2.1 in [3])
Assume a non-negative stock price $S = (S_t)_t$ with $t \in [0,\infty)$ has non-negative random dividends $\Delta = (\Delta_t)_t$ and drift $\mu = (\mu_t)_t$.

Let $R_t := \exp(\int_0^t \mu_s\, ds)$ and define the discounted stock price process as $\tilde{S}_t := S_t/R_t$ and the discounted dividends as $\tilde{\Delta}_t := \Delta_t/R_{t_j}$. We also denote $\tilde{\Delta}_t := \mathbb{E}[\tilde{\Delta}_t]$ and define $S_0^* = S_0 - \int_0^\infty \tilde{\Delta}_s\, ds$, which we assume to be strictly positive.[5] *Then,*

1. *There exists a non-negative (local) martingale Z such that*

$$\tilde{S}_t = S_0^* Z_t - \int_0^t \tilde{\Delta}_s\, ds \tag{18}$$

2. *There exist a non-negative (local) martingale X such that*

$$\tilde{S}_t = S_0^* X_t + \int_t^\infty \mathbb{E}_t[\tilde{\Delta}_s]\, ds\ . \tag{19}$$

3. *The two representations are related by*

$$Z_t = X_t + \frac{1}{S_0^*}\int_0^\infty \mathbb{E}_t[\tilde{\Delta}_s]\, ds\ . \tag{20}$$

Corollary 5.1 (Reverse statement to theorem 5.1) *For any non-negative local martingale X, the stock price S defined by (19) is valid, i.e. pays dividends Δ and is non-negative.*

The same statement is not true for Z, i.e. not every non-negative Z defines via (18) a non-negative stock price process.

[5] S_0^* must be non-negative by no-arbitrage.

Following (18), we easily see in terms of Z the SDE for S has the natural dynamics

$$d\tilde{S}_t = S_0^* dZ_t - \tilde{\Delta}_t \, dt \ .$$

In terms of X, we have the "consistent" dynamics

$$d\tilde{S}_t = S_0^* dX_t - \tilde{\Delta}_t \, dt + \int_t^\infty d\mathbb{E}_t[\tilde{\Delta}_s] \, ds \ .$$

Corollary 5.2 (Generalization of Proposition 2.1 in [3]) *Equation (19) implies*

$$\tilde{S}_t \geq \int_t^\infty \mathbb{E}_t[\tilde{\Delta}_s] \, ds \ .$$

Corollary 5.3 (Cash Dividends) *Assume that dividends are cash, i.e.* $\Delta_t = d_t$ *fixed. Then we obtain the result shown in [3] that we can write*

$$\tilde{S}_t = (\tilde{F}_t - \tilde{D}_t) X_t + \tilde{D}_t$$

where $\tilde{D}_t = \int_t^\infty \tilde{d}_s \, ds$.

Proof of the theorem– the discounted total return process in (18),

$$S_0^* Z_t := \tilde{S}_t + \int_0^t \tilde{\Delta}_s \, ds \ ,$$

represents the (non-negative) re-invested equity process, and is therefore tradable in the mathematical sense. Under no-arbitrage in the sense of "no free lunch without vanishing risk" as defined in [5], this implies that Z is a (local) martingale.

Define X as in (20). Then,

$$S_0^* X_t = S_0^* Z_t - \int_0^t \tilde{\Delta}_s \, ds - \int_t^\infty \mathbb{E}_t[\tilde{\Delta}_s] \, dt = \tilde{S}_t - \int_t^\infty \mathbb{E}_t[\tilde{\Delta}_s] \, ds \ ,$$

which shows (19).

It remains to show that X is non-negative: to this end, assume that $(T_i)_i$ is an increasing sequence of finite stopping times which localizes Z over $[0, \infty)$. We use $a \vee b := \max(a, b)$.

$$0 \leq \mathbb{E}_t[\tilde{S}_{T_i \vee t}] = \tilde{S}_t + \mathbb{E}_t[\tilde{S}_{T_i \vee t} - \tilde{S}_t] \stackrel{(18)}{=} \tilde{S}_t - \int_t^{T_i \vee t} \mathbb{E}_t[\tilde{\Delta}_s] \, ds =: S_0^* X_t^{(i)} \ ,$$

i.e. $X_t^{(i)} \downarrow X_t$, which implies $X_t \geq 0$ by monotone convergence. $\square$

6 Summary

We have presented a modelling approach for consistent dividends which provides us with implied short-term "cash" dividends and long-term "proportional" dividends consistently across time, conditional on the state of the market. To our knowledge, this is the first proposal in this direction at the time this was presented at Global Derivatives 2012 [4].

Our approach is motivated by the general representation for dividend paying stocks which we have proved in wide generality with theorem 5.1. This extends the results in [3] to the general case of stochastic dividends in a local martingale setting.

When using the mean-variance affine framework presented here, we have closed forms for spot, conditional forwards and dividend swaps which makes the model numerically efficient to simulate.

In this note we have not discussed how to calibrate the model. As an indication: we recommend calibrating the dividend volatility and mean-reversion parameters to historic data, with standard valuation checking to ensure we are marked well vs. the market. The local volatility parameter for the equity is then calibrated using a classic bootstrap "Dupire"-like calibration, for example using the "particle method" presented by Guyon/Labrodere in their seminal article [7].

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